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Exact solutions of coupled scalar field equations

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Abstract. Exact analytical solutions of a generic system of coupled ordinary differential equations for a pair of real scalar fields have been explicitly obtained. The equations may be considered to be the stationary form of the coupled time-dependent Schrödinger-Boussinesq (or Korteweg-de Vries) equations. The generic equations have six free parameters whereas it has been possible to obtain exact solutions valid on a five-dimensional hypersurface in six-dimensional parameter space using a technique developed by Varma and Rao. While the solution for the variable of the Boussinesq (or Korteweg-de Vries) equation always has a symmetric structure, both symmetric and antisymmetric solutions are possible for the variable of the Schrödinger equation. The results are applied to an example dealing with the stationary propagation of coupled non-linear upper-hybrid and magnetosonic waves in magnetised plasmas.

1. Introduction

Coupled second-order ordinary differential equations for a pair of real scalar fields occur in many branches of physics. For example, in particle physics, the relativistic quantum field theories in $1+1$ dimensions for finite-energy localised fields lead to a set of non-linear equations for a pair of scalar fields, say ρ and σ [1]. On the other hand, in plasma physics, the non-linear development of the instability associated with the envelope of an high-frequency wave field (E) coupled to an appropriate low-frequency wave field (ϕ) is governed by a pair of coupled equations [2]. An outstanding mathematical problem associated with such non-linear equations is to obtain their exact analytical solutions valid in the entire allowed space spanned by the free parameters occurring in the equations. In the case of (ρ, σ) fields, the relevant equations have three free parameters whereas it has been possible to obtain their exact solutions only for some specific numerical values of some of the parameters [3] or, at the most, with two of the parameters being free, in which case the third parameter is self-consistently determined [4]. On the other hand, when E and ϕ correspond, respectively, to the Langmuir and ion-acoustic wave fields, the governing equations have two free parameters and Nishikawa *et al* [5] obtained exact localised solutions valid on a straight line in the two-dimensional parameter space. Similar governing equations, but with more free parameters, occur in other problems dealing with modulational instability of, for instance, coupled electromagnetic-ion acoustic waves [6], upper hybrid-magnetosonic waves [7], etc. It is, therefore, necessary to obtain the exact solutions of a generic system of equations with as many free parameters as possible. Such a generic system for the (ρ, σ) fields [4] consists of eight free parameters whereas the equations for the (E, ϕ) fields [6] contain seven free parameters. However, it has not been possible so far to obtain the exact solutions in either case.

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We investigate, in this paper, the existence of analytic solutions for a generic system of equations for the E and ϕ fields and, in particular, obtain their exact solutions having as many as five parameters free. These equations occur in all modulational problems in plasma physics when the dynamics of the low-frequency motion is described in terms of a non-linear equation like the (driven) Boussinesq equation (for bidirectional propagation) or the (driven) Korteweg-de Vries equation (for unidirectional propagation). In the stationary frame, these equations give rise to the equation for the low-frequency field (ϕ). On the other hand, the equation for the high-frequency field envelope (E) is obtained from a Schrödinger-like equation where the low-frequency field occurs as a potential [2]. The coupling between the two equations arises due to a term corresponding to the (non-linear) ponderomotive force [8] of the high-frequency wave field.

The equations are solved using a method of solution developed by Varma and Rao [9] (see Rao and Varma [10] for more details). The method, as it exists, is applicable to any pair of second-order equations not containing terms involving the first derivatives of the dependent variables but admitting a 'Hamiltonian' which is a constant of motion. The equations can be thought of as the equations of motion for a classical particle in a two-dimensional conservative potential which, for complete integrability, requires the existence of two constants of motion. The search for the second constant of motion is made easier by obtaining a governing equation for the trajectory of the particle in the (E, ϕ) space where the independent variable occurs only as a parameter. In the case of the generic equations considered below, it turns out that the second constant of motion is just a quadratic polynomial in ϕ for E^2 which enables us to obtain the exact solutions for E and ϕ.

The paper is organised as follows. In the next section, we write down the generic equations and discuss their properties. We obtain in § 3 different types of exact solutions which are valid for different parameter regimes. Some special cases of the generic equations are discussed in § 4. Section 5 is devoted to an application of these results to the Schrödinger-Boussinesq (or Korteweg-de Vries) system that governs the (stationary) propagation of coupled upper-hybrid and magnetosonic waves in magnetised plasmas. A summary of the results and some general remarks are given in § 6.

2. Generic equations

Consider the system of equations

$$\beta \frac{d^2 E}{d\xi^2} = b_1 E + b_2 \phi E \tag{1}$$

$$\lambda \frac{d^2 \phi}{d\xi^2} = d_1 \phi + d_2 \phi^2 + d_3 E^2 \tag{2}$$

where E and ϕ are real fields, ξ is the (real) independent variable and all the other remaining quantities are free parameters. While the parameter β can be taken, without any loss of generality, to be equal to unity, we explicitly retain the parameter λ so that the singular case $\lambda = 0$ can be easily considered. Thus, the system of equations has six free parameters. (A derivation of (1) and (2) from the coupled time-dependent Schrödinger-Boussinesq (or Korteweg-de Vries) system is given in the appendix.) By inspection, it follows that the equations are invariant under the transformations (i)

$\xi \to -\xi$, (ii) $E \to -E$, and (iii) $\xi \to \xi + C$, where C is a constant. Furthermore, the equations can be derived from a Lagrangian

$$L = \beta d_3 \left(\frac{\mathrm{d}E}{\mathrm{d}\xi}\right)^2 + \tfrac{1}{2}\lambda b_2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 - V(E, \phi) \tag{3}$$

where,

$$V(E, \phi) = -(d_3 b_1 E^2 + \tfrac{1}{2} b_2 d_1 \phi^2 + \tfrac{1}{3} b_2 d_2 \phi^3 + b_2 d_3 \phi E^2) \tag{4}$$

represents the two-dimensional effective potential for the motion of a classical particle with two degrees of freedom, namely E and ϕ when ξ is treated as the time variable. If π_E and π_ϕ denote, respectively, the conjugate momenta corresponding to E and ϕ, we obtain

$$\pi_E = 2\beta d_3 \left(\frac{\mathrm{d}E}{\mathrm{d}\xi}\right) \qquad \pi_\phi = \lambda b_2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) \tag{5}$$

and the Hamiltonian

$$H = \frac{1}{4\beta d_3} (\pi_E)^2 + \frac{1}{2\lambda b_2} (\pi_\phi)^2 + V(E, \phi) \tag{6}$$

which is an 'integral of motion' since the 'time' variable (ξ) does not explicitly appear in the Lagrangian (L).

It is well known [11] in classical dynamics that a system with two degrees of freedom is completely integrable if it possesses two integrals of motion which are in involution with each other. In the present case, this requires the existence of another integral of motion, in addition to the Hamiltonian (H). In the next section, we look for the second integral by deriving an equation for the trajectory of the system in the (E, ϕ) space.

3. Exact analytical solutions

The independent variable ξ in (1) and (2) can be eliminated using (6) to yield an equation for E in terms of ϕ only [9, 10]. Since (1) and (2) are invariant under $E \to -E$, we define a new variable $\psi = E^2$ and obtain

$$4\lambda\beta\psi(H - V)\frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} + \beta^2 d_3 h(\psi, \phi)\left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right)^3 - 2\lambda\beta[d_3 g(\psi, \phi) + H - V]\left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right)^2$$
$$+ 2\lambda\beta b_2 \psi h(\psi, \phi)\left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right) - 4\lambda^2 b_2 \psi g(\psi, \phi) = 0 \tag{7}$$

where

$$h(\psi, \phi) = d_1\phi + d_2\phi^2 + d_3\psi \tag{8}$$

$$g(\psi, \phi) = b_1\psi + b_2\phi\psi \tag{9}$$

$$V(\psi, \phi) = -(d_3 b_1 \psi + \tfrac{1}{2} b_2 d_1 \phi^2 + \tfrac{1}{3} b_2 d_2 \phi^3 + b_2 d_3 \phi\psi). \tag{10}$$

Equation (7) determines the evolution of the system in the (ψ, ϕ) space with ξ acting as a parameter along the trajectories. Alternatively, any solution of (7) is also an implicit solution of the generic equations (1) and (2), and is an integral of motion for the coupled equations.

Since the variable ϕ occurs as a polynomial in the coefficients of (7), we seek a solution of the form

$$\psi = \sum_{n=0}^{\infty} a_n \phi^n \tag{11}$$

where the coefficients a_n which are functions of the free parameters, are to be determined self-consistently. To this end, we substitute (11) into (7) and equate the coefficients of like powers of ϕ to zero. For localised boundary conditions

$$E, \phi, \frac{dE}{d\xi}, \frac{d\phi}{d\xi} \to 0 \qquad \text{as} \qquad \xi \to \pm\infty \tag{12}$$

the coefficient a_0 is identically equal to zero. On the other hand, all the higher coefficients (a_n, $n > 0$) are uniquely and explicitly determined in terms of the free parameters as solutions of first-order algebraic equations for every n. In particular, the coefficients a_1 and a_2 are given by

$$a_1 = \frac{1}{\beta d_3}(4\lambda b_1 - \beta d_1) \tag{13}$$

$$a_2 = a_1 \frac{(3\lambda b_2 - \beta d_2)}{(16\lambda b_1 - \beta d_1)}. \tag{14}$$

Similarly, the remaining coefficients ($n > 2$) can be easily obtained.

However, for parameter values satisfying the relation

$$\beta d_1 b_2 - 2b_1(3\beta d_2 - \lambda b_2) = 0 \tag{15}$$

it can be shown that (i) all the coefficients a_n, $n \geq 3$, are identically zero, and (ii) the coefficient of each power of ϕ in (7) is also zero. This implies that the relation

$$\psi = a_1\phi + a_2\phi^2 \tag{16}$$

is an exact solution of the trajectory (7). The coefficients a_1 and a_2 given by (13) and (14) can be simplified using (15) to yield

$$a_1 = \frac{6b_1}{\beta d_3 b_2}(\lambda b_2 - \beta d_2) \tag{17}$$

$$a_2 = \frac{b_2}{6b_1} a_1. \tag{18}$$

The quantity in (16), namely

$$G(E, \phi) \equiv E^2 - a_1\phi - a_2\phi^2 \tag{19}$$

can be considered as an integral of motion for the coupled system of (1) and (2). Furthermore, the Poisson bracket between G and H, namely

$$[G, H] = \sum_{x=E,\phi} \left(\frac{\partial G}{\partial x}\frac{\partial H}{\partial \pi_x} - \frac{\partial H}{\partial x}\frac{\partial G}{\partial \pi_x}\right) \tag{20}$$

is identically zero, implying that the two integrals are in involution. This proves the complete integrability of the generic system (1) and (2) for the case when the free parameters satisfy the supplementary relation (15).

Explicit solutions for $E(\xi)$ and $\phi(\xi)$ can be readily obtained by using (16) in (2) which becomes

$$\lambda\, d^2\phi/d\xi^2=(d_1+d_3a_1)\phi+(d_2+d_3a_2)\phi^2. \tag{21}$$

For localised solutions satisfying the boundary conditions (12), one obtains

$$\phi(\xi)=\phi_0\,\mathrm{sech}^2[\mu(\xi-\xi_0)] \tag{22}$$

where $\phi_0=-6b_1/b_2$, $\mu=(b_1/\beta)^{1/2}$ and ξ_0 is a constant of integration. The solution for $E(\xi)$ is obtained from (16) and (22) as

$$E(\xi)=\pm E_0\,\mathrm{sech}[\mu(\xi-\xi_0)]\tanh[\mu(\xi-\xi_0)] \tag{23}$$

where

$$E_0=\frac{6b_1}{b_2}\left(\frac{1}{\beta d_3}(\beta d_2-\lambda b_2)\right)^{1/2}. \tag{24}$$

In order that the solutions are localised, it is necessary that the parameters satisfy the inequalities

$$\begin{gathered} b_1\beta>0 \\ \beta d_3(\beta d_2-\lambda b_2)>0. \end{gathered} \tag{25}$$

Clearly, while the solution $E(\xi)$ is antisymmetric with respect to $\xi=\xi_0$ for any values of the parameters, the solution $\phi(\xi)$ always has a symmetric structure.

The solutions (22) and (23) contain five free parameters. We have thus obtained exact localised solutions of (1) and (2) valid on a five-dimensional hypersurface in the six-dimensional parameter space. The solutions reported earlier for coupled Langmuir-ion acoustic [5], electromagnetic-ion acoustic [6] and upper hybrid-magnetosonic [7] waves follow directly as special cases of the solutions (22) and (23).

The basic equations (1) and (2) admit a new class of symmetric solutions for $E(\xi)$ as well as $\phi(\xi)$ which, to our knowledge, have not been reported in the literature earlier. Following the procedure outlined above, it can be shown that, for parameter values satisfying the equation

$$3\lambda b_2-\beta d_2=0 \tag{26}$$

all the coefficients a_n in (11) are identically zero for $n>1$ with a_1 given by (13). Furthermore, the coefficients of successive powers of ϕ in (7) are also exactly balanced. The remaining steps to construct the explicit solutions are straightforward, resulting in the solutions

$$E(\xi)=\pm E_0\,\mathrm{sech}[\mu(\xi-\xi_0)] \tag{27}$$

$$\phi(\xi)=\phi_0\,\mathrm{sech}^2[\mu(\xi-\xi_0)] \tag{28}$$

where

$$\begin{gathered} E_0=\left(\frac{6\lambda b_1}{\beta^2 d_2 d_3}(\beta d_1-4\lambda b_1)\right)^{1/2} \\ \phi_0=-2b_1/b_2. \end{gathered} \tag{29}$$

In order that the solutions are consistent with the localised boundary conditions (12), the various parameters should satisfy

$$\begin{gathered} b_1\beta>0 \\ b_2d_3(\beta d_1-4\lambda b_1)>0. \end{gathered} \tag{30}$$

In the following, we consider some special cases of (1) and (2) which are often encountered in the study of envelope waves in plasmas.

4. Some special cases

4.1. $d_2 = 0$

In this case, (2) reduces to

$$\lambda \, \mathrm{d}^2\phi/\mathrm{d}\xi^2 = d_1\phi + d_3E^2 \tag{31}$$

which is a linear equation in ϕ but is coupled to (1) through the non-linear term d_3E^2. The solutions of (1) and (31) are obtained from (22) and (23) as

$$E(\xi) = \pm\left(\frac{18b_1d_1}{d_3b_2}\right)^{1/2} \mathrm{sech}[\mu(\xi-\xi_0)] \tanh[\mu(\xi-\xi_0)] \tag{32}$$

$$\phi(\xi) = \frac{3\beta d_1}{\lambda b_2} \mathrm{sech}^2[\mu(\xi-\xi_0)] \tag{33}$$

where the various parameters satisfy the supplementary relation:

$$\beta d_1 + 2\lambda b_1 = 0. \tag{34}$$

Solutions (32) and (33) are structurally similar to those obtained in the general case when $d_2 \neq 0$. Thus, it follows that the ϕ non-linearity in (2) is not essential in order to have antisymmetric solutions for the E field. (It may be noted that this is contrary to the commonly held view that such solutions arise solely due to the presence of the ϕ^2 term in (2) [5, 12].

4.2. $\lambda = 0$

For this case, (2) becomes an algebraic equation:

$$d_2\phi^2 + d_1\phi + d_3E^2 = 0 \tag{35}$$

which is coupled to (1). However, it turns out that the coefficients a_1 and a_2, as given by (13) and (14), are still valid even though in the limit $\lambda = 0$, (2) is singular! For, using the expressions for a_1 and a_2 with $\lambda = 0$ in (11), one obtains just (35). Solving (35) for ϕ in terms of E^2

$$\phi = \frac{1}{2d_2}[-d_1 + p(d_1^2 - 4d_2d_3E^2)^{1/2}] \qquad p = \pm 1$$

which, together with (1), leads to

$$\left(\frac{\mathrm{d}E}{\mathrm{d}\xi}\right)^2 = -\frac{(2d_2b_1 - b_2d_1)}{8d_2^2d_3\beta}(d_1^2 - 4d_2d_3E^2)[1 + p\eta(d_1^2 - 4d_2d_3E^2)^{1/2}] \tag{36}$$

where the constant of integration has been evaluated using the condition that $\mathrm{d}E/\mathrm{d}\xi = 0$ whenever $E^2 = d_1^2/4d_2d_3$, and

$$\eta = \frac{2b_2}{3(2b_1d_2 - b_2d_1)}. \tag{37}$$

Equation (36) can easily be written in a standard form which admits solutions in terms of Jacobian elliptic functions [13].

However, particularly simple solutions exist for $\eta d_1 = \pm 1$, namely

$$E = \pm E_0 \operatorname{sech}[\mu(\xi - \xi_0)] \tanh[\mu(\xi - \xi_0)] \tag{38}$$

where

$$E_0 = \frac{d_1}{(d_2 d_3)^{1/2}} \qquad \mu = \left(-\frac{b_2}{6\eta\beta d_2}\right)^{1/2}.$$

Both the solutions (38) exist for either value of ηd_1. For $\eta d_1 = -1$, i.e. $b_2 d_1 = 6 b_1 d_2$, $\phi(\xi)$ is obtained as

$$\phi(\xi) = -\frac{d_1}{d_2} \operatorname{sech}^2[(b_1/\beta)^{1/2}(\xi - \xi_0)]. \tag{39}$$

It may be noted that the solutions (38) and (39) follow also from the solutions (22) and (23) in the limit $\lambda = 0$ even though, as mentioned earlier, (2) is singular in that limit. On the other hand, for $\eta d_1 = +1$, i.e. $5 b_2 d_1 = 6 b_1 d_2$, $\phi(\xi)$ is given by

$$\phi(\xi) = -\frac{d_1}{d_2} \tanh^2[(-b_1/5\beta)^{1/2}(\xi - \xi_0)]. \tag{40}$$

Both the solutions (39) and (40) are symmetric with respect to $\xi = \xi_0$.

In the next section, we apply the above results to the case of coupled Schrödinger–Boussinesq equations which govern the (stationary) propagation of non-linear upper-hybrid and magnetosonic waves in a magnetised plasma.

5. An example: coupled Schrödinger–Boussinesq equations

5.1. Basic equations

The bidirectional near-magnetosonic propagation of upper-hybrid waves of frequency ω_0 and wavenumber k_0 coupled to the magnetosonic waves in a homogeneous magnetised plasma is governed [7] by a Schrödinger-like equation, namely

$$\mathrm{i}\left(\frac{\partial E}{\partial t} + V_g \frac{\partial E}{\partial x}\right) + \tfrac{1}{2} D_0 \frac{\partial^2 E}{\partial x^2} = -\tilde{\lambda} E + \omega_{H0} \mu N E \tag{41}$$

which is coupled to the (driven) Boussinesq equation:

$$\frac{\partial^2 N}{\partial t^2} - V_M^2 \frac{\partial^2 N}{\partial x^2} - \beta^2 \frac{\partial^4 N}{\partial x^4} - a^2 \frac{\partial^2}{\partial x^2}(N^2) = \eta^2 \frac{\partial^2}{\partial x^2}(E E^*). \tag{42}$$

In (41) and (42), $E(x, t)$ is the complex envelope of the upper-hybrid wave electric field normalised with respect to $(16\pi n_0 T_e)^{1/2}$, N is the perturbed number density (in the low-frequency response) normalised with respect to n_0, x and t are, respectively, the space and time variables, and the asterisk denotes complex conjugation. The following notations are used in (1) and (2): $V_g = k_0 D_0$, $D_0 = 3\omega_{pe0}^2 v_{te}^2 (\omega_{pe0}^2 - 3\Omega_{e0}^2)^{-1}/\omega_{H0}$, $\tilde{\lambda} = \omega_0 - \omega_{H0} - D_0 k_0^2/2$, $\omega_{H0}^2 = \omega_{pe0}^2 + \Omega_{e0}^2$, $\mu = (\frac{1}{2}\omega_{pe0}^2 + \Omega_{e0}^2)/(\omega_{pe0}^2 + \Omega_{e0}^2)$, $V_M^2 = V_A^2 + C_s^2$, $\beta = V_A c/\omega_{pe0}$, $a^2 = (3V_A^2 + 2C_s^2)/2$ and $\eta = \omega_{H0} C_s/\omega_{pe0}$. The remaining notations are standard [7] and the subscript '0' denotes equilibrium quantities. In the linear limit without modulations, (41) and (42) give, respectively, the well known dispersion relations for the upper-hybrid and magnetosonic waves in the long wavelength limit [8, 14].

Looking for stationary solutions for E and N, we define (cf the appendix)

$$E(x, t) = E(\xi) \quad \exp[\mathrm{i}\{X(x) + T(t)\}]$$
$$N(x, t) = N(\xi) \tag{43}$$

where $\xi = x - Mt$ and $E(\xi)$ is real. The parameter M determines the speed of the stationary envelope wave. Using (13) in (41) and (42), one obtains

$$D_0 \, \mathrm{d}^2E/\mathrm{d}\xi^2 = \lambda E + 2\mu\omega_{\mathrm{H}0} NE \tag{44}$$

where $\lambda = 2\delta - 2\tilde{\lambda} + (M^2 - V_g^2)/D_0$ accounts for the total non-linear shift in frequency and wavenumber, $\delta = \mathrm{d}T/\mathrm{d}t$ denotes the frequency shift, and

$$\beta^2 \, \mathrm{d}^2N/\mathrm{d}\xi^2 = -(V_M^2 - M^2)N - a^2N^2 - \eta^2E^2 \tag{45}$$

which is coupled to (44) by the last term.

5.2. Exact solutions

Equations (44) and (45) are in the standard form (cf (1) and (2)). Using the results of § 3, their exact localised solutions satisfying boundary conditions similar to (12) can easily be obtained. For the case when the various parameters satisfy the relation

$$V_M^2 - M^2 = \frac{\lambda}{\mu\omega_{\mathrm{H}0}}\left(3a^2 + \frac{2\beta^2\mu\omega_{\mathrm{H}0}}{D_0}\right) \tag{46}$$

the solutions are

$$E(\xi) = \pm E_0 \operatorname{sech}(\kappa\xi)\tanh(\kappa\xi) \tag{47}$$

$$N(\xi) = -N_0 \operatorname{sech}^2(\kappa\xi) \tag{48}$$

where

$$E_0 = \left(\frac{3\lambda}{\eta\mu\omega_{\mathrm{H}0}}\right)\left(a^2 + \frac{2\beta^2\mu\omega_{\mathrm{H}0}}{D_0}\right)^{1/2}$$
$$N_0 = \frac{3\lambda}{\mu\omega_{\mathrm{H}0}} \qquad \kappa = (\lambda/D_0)^{1/2} \tag{49}$$

and the constant of integration ξ_0 is taken, without any loss of generality, to be zero. The total electric field is then given by

$$E(x, t) = E(\xi)\exp[\mathrm{i}(\nu x + \delta t)] \tag{50}$$

where $\nu = (M - V_g)/D_0$.

Clearly, while the wave envelope $E(\xi)$ is antisymmetric with respect to the centre ($\xi = 0$), the perturbed number density $N(\xi)$ has a symmetric structure. Equation (46) indicates that both super-magnetosonic ($M > V_M$) as well as sub-magnetosonic ($M < V_M$) solitons are possible. Since κ should be real, it follows that for positive (negative) dispersion of upper-hybrid waves, i.e. $D > 0$ ($D < 0$), the solitons are associated with rarefaction (compressional) density perturbations moving with sub-magnetosonic (super-magnetosonic) speeds.

The class of symmetric solutions for both $E(\xi)$ and $N(\xi)$ are obtained for parameter values such that

$$6\beta^2\mu\omega_{H0} + D_0 a^2 = 0. \tag{51}$$

The exact solutions in this case are

$$E(\xi) = \pm E_0 \operatorname{sech}(\kappa\xi) \tag{52}$$

$$N(\xi) = N_0 \operatorname{sech}^2(\kappa\xi) \tag{53}$$

where

$$E_0 = \frac{(6\lambda)^{1/2}\beta}{D_0 a\eta}[D_0(M^2 - V_M^2) - 4\lambda\beta^2]^{1/2}$$
$$N_0 = -\frac{\lambda}{\mu\omega_{H0}} \qquad \kappa = (\lambda/D_0)^{1/2}. \tag{54}$$

Since (51) requires $D_0 < 0$, κ is real only for $\lambda < 0$. This implies that, unlike the previous case, only compressional ($N > 0$) density perturbations are possible.

The coupled equations (44) and (45) admit, as pointed out for the generic equations, antisymmetric solutions for $E(\xi)$ even when the non-linear term in N on the right-hand side of (45) does not exist, i.e. when the low-frequency (magnetosonic) dynamics is governed by a linear Boussinesq equation driven by the non-linear ponderomotive force. The corresponding results are obtained from (46)-(49) by formally letting the parameter a to be equal to zero. Clearly, the solutions for $E(\xi)$ and $N(\xi)$ do not change qualitatively. The amplitude E_0 is given in the present case by

$$E_0 = \frac{3\sqrt{2}\,\lambda\beta}{\eta(\mu\omega_{H0}D_0)^{1/2}} \tag{55}$$

which requires $D_0 > 0$. From (49), it follows that $\lambda > 0$ and $N_0 < 0$ which, together with (46), yield $M^2 < V_M^2$. Comparing these results with the results following from (46)-(49), we conclude that the N^2 non-linear term in (45) is responsible for driving the compressional ($N > 0$) and super-magnetosonic ($M > V_M$) solitons of coupled upper hybrid-magnetosonic waves.

We finally consider the case when the low-frequency dispersion term in (45) is absent which is formally the same as putting $\beta = 0$. There are two different cases.

(i) When $V_M^2 - M^2 = 3\lambda a^2/\mu\omega_{H0}$ the solutions are given by (47) and (48), with

$$E_0 = \left(-\frac{3\lambda}{\eta^2\mu\omega_{H0}}\right)^{1/2}. \tag{56}$$

The quantities N_0 and κ are as in (49). The solitons in this case are compressional ($N > 0$) and move with super-magnetosonic speeds.

(ii) When $V_M^2 - M^2 = 6\lambda a^2/10\mu\omega_{H0}$ the solution for $E(\xi)$ is similar to that in the previous case with

$$E_0 = \left(-\frac{6\lambda}{5\eta^2\mu\omega_{H0}}\right)^{1/2}. \tag{57}$$

However, $N(\xi)$ is qualitatively different from (48):

$$N(\xi) = -\frac{6\lambda}{5\mu\omega_{H0}}\tanh^2(\kappa\xi) \tag{58}$$

where $\kappa = (-\lambda/5D_0)^{1/2}$.

5.3. *Integral invariants*

The time-dependent equations (41) and (42) admit a set of four exact integral invariants which can be explicitly evaluated for the different solutions obtained above. The invariants can be derived using the Lagrangian density

$$\mathscr{L} = -\frac{1}{\mu\omega_{H0}}[\tfrac{1}{2}i(E^*E_t - EE_t^*) + \tfrac{1}{2}iV_g(E^*E_x - EE_x^*) - \tfrac{1}{2}D_0E_xE_x^* + \tilde{\lambda}EE^*] + \frac{1}{\eta^2}[\tfrac{1}{2}\phi^2 + \theta\phi_t + \tfrac{1}{2}V_M^2(\theta_x)^2 + \tfrac{1}{3}a^2(\theta_x)^3 + \beta^2\theta_x\psi_x + \tfrac{1}{2}\beta^2\psi^2] + \theta_xEE^* \tag{59}$$

where θ, ψ and ϕ are auxiliary variables, and θ_x is to be identified as N; the subscripts x and t denote, respectively, the partial derivatives. The corresponding Hamiltonian ($\mathscr{H}$) and momentum (π) densities are given by

$$\mathscr{H} = \frac{iV_g}{2\mu\omega_{H0}}(E^*E_x - EE_x^*) - \frac{D_0}{2\mu\omega_{H0}}E_xE_x^* + \frac{\tilde{\lambda}}{\mu\omega_{H0}}EE^* - \frac{1}{\eta^2}[\tfrac{1}{2}\phi^2 + \tfrac{1}{2}V_M^2N^2 + \tfrac{1}{3}a^2N^3 + \beta^2NN_{xx} + \tfrac{1}{2}\beta^2(N_x)^2] - NEE^* \tag{60}$$

$$\pi = \frac{i}{2\mu\omega_{H0}}(EE_x^* - E^*E_x) + \frac{1}{\eta^2}\theta N_t \tag{61}$$

where $\phi = \theta_t$. The integral invariants are then defined by

$$I_H = \int_{-\infty}^{+\infty} \mathscr{H}\,dx \qquad I_\pi = \int_{-\infty}^{+\infty} \pi\,dx$$
$$I_E = \int_{-\infty}^{+\infty} EE^*\,dx \qquad I_N = \int_{-\infty}^{+\infty} N_t\,dx \tag{62}$$

where I_N is obtained directly from (42).

The invariants (62) can be explicitly evaluated for any of the exact solutions obtained earlier. However, we present below the analysis for the solutions (47) and (48) which can easily be extended to the other cases. The algebra involved, though straightforward, is quite cumbersome but can be simplified drastically using the following procedure. Using (46)-(50), the integrands in (62) can be written in terms of powers of N only:

$$\mathscr{H} = \mathscr{H}_1N + \mathscr{H}_2N^2 + \mathscr{H}_3N^3$$
$$\pi = \pi_1N + \pi_2N^2$$
$$EE^* = -\left(\frac{E_0^2}{N_0}\right)N - \left(\frac{E_0}{N_0}\right)^2N^2 \tag{63}$$

where the coefficients are defined by

$$\mathscr{H}_1 = \frac{E_0^2}{2\mu\omega_{H0}N_0}(2\nu V_g + \nu^2D_0 - 2\tilde{\lambda} + \lambda)$$
$$\mathscr{H}_2 = \frac{E_0^2}{2\mu\omega_{H0}N_0^2}(2\nu V_g + \nu^2D_0 - 2\tilde{\lambda} + 4\lambda) + E_0^2/N_0 - \frac{1}{\eta^2}[\tfrac{1}{2}(M^2 + V_M^2) + 6\beta^2\kappa^2]$$
$$\mathscr{H}_3 = \frac{2\lambda E_0^2}{\mu\omega_{H0}N_0^3} + \left(\frac{E_0}{N_0}\right)^2 - \frac{1}{\eta^2}(\tfrac{1}{3}a^2 + 8\beta^2\kappa^2/N_0) \tag{64}$$
$$\pi_1 = -\left(\frac{\nu E_0^2}{\mu\omega_{H0}N_0} + \frac{2MN_0}{\eta^2}\right) \qquad \pi_2 = \pi_1/N_0.$$

Thus it is sufficient to evaluate the integrals of powers of N only. This can further be simplified by defining the definite integral:

$$I^{(j)}=\int_{-\infty}^{+\infty} N^j \,\mathrm{d}x \tag{65}$$

where j is a positive integer greater than zero. Substituting the solution (48) into (65), one easily obtains the recurrence relation:

$$I^{(j+1)}=-\frac{2jN_0}{(2j+1)}I^{(j)} \tag{66}$$

where $I^{(1)}=-2N_0/\kappa$. Using (66), we get $I^{(2)}=4N_0^2/3\kappa$ and $I^{(3)}=-16N_0^3/15\kappa$. Combining (62), (63) and (65), we finally obtain the invariants

$$I_{\mathscr{H}}=-\frac{2N_0}{15\kappa}(15\mathscr{H}_1-10N_0\mathscr{H}_2+8N_0^2\mathscr{H}_3) \tag{67}$$

$$I_\pi=-2N_0\pi_1/3\kappa \tag{68}$$

$$I_E=2E_0^2/3\kappa. \tag{69}$$

By direct integration, one finds that the invariant $I_N=0$. This is consistent with the fact that $\int_{-\infty}^{+\infty} N_t\,\mathrm{d}x$ which represents the net rate of change of number density over the entire range should be zero as there are no sources or sinks in the system.

6. Summary and remarks

The main results of the present work can be summarised as follows. Starting with a generic system of coupled non-linear ordinary differential equations, we have obtained their exact solutions containing as many as five free parameters. The problem is identical with the motion of a classical particle in a two-dimensional conservative potential and our results establish the integrability of the system for a class of initial/boundary conditions. The generic equations are derivable from the time-dependent Schrödinger–Boussinesq (or Korteweg–de Vries) system which governs the non-linear development of the modulational instability of a class of high-frequency waves (like Langmuir, electromagnetic, upper-hybrid waves) coupled to appropriate low-frequency waves (like ion-acoustic, magnetosonic waves) in plasmas. As an illistration, we have considered the case of coupled upper-hybrid and magnetosonic waves in magnetised plasmas and have shown the existence of different classes of exact solutions for different ranges of parameters. We have also explicitly evaluated a set of four exact integral invariants for the relevant time-dependent equations.

Finally, we make a few remarks about some of the questions that need further investigations. Firstly, the problem of obtaining the exact solutions valid in the entire parameter space is still open. Even the existence of such solutions has not been discussed so far. It is possible that strictly localised solutions exist only for some specific regions of the parameter space. Secondly, it would be interesting to look for exact solutions corresponding to more general initial/boundary conditions, for example, periodic solutions valid over larger parameter space than considered above. Thirdly, the method of solution used here can be tried with necessary extensions to other coupled equations like those found in relativistic quantum field theories [1]. Fourthly, the stability [15, 16] as well as the interaction [17, 18] of the stationary solutions of the coupled Schrödinger-Boussinesq (or Korteweg-de Vries) system can now be investigated using the exact invariants explicitly obtained in the present analysis.

Appendix. Derivation of the generic equations from the Schrödinger–Boussinesq (or Korteweg–de Vries) system of equations

As pointed out in § 5, the generic equations (1) and (2) can be seen as the stationary form of the coupled Schrödinger-Boussinesq (or Korteweg-de Vries) system. We present below the relevant details.

Consider first the case when the Schrödinger equation is coupled to the Boussinesq equation:

$$\mathrm{i}\left(\frac{\partial E}{\partial t}+\lambda_1\frac{\partial E}{\partial x}\right)+\lambda_2\frac{\partial^2 E}{\partial x^2}=\lambda_3 E+\lambda_4 NE \tag{A1}$$

$$\frac{\partial^2 N}{\partial t^2}+\mu_1\frac{\partial^2 N}{\partial x^2}+\mu_2\frac{\partial^4 N}{\partial x^4}+\mu_3\frac{\partial^2(N^2)}{\partial x^2}=\mu_4\frac{\partial^2(EE^*)}{\partial x^2} \tag{A2}$$

where E is a complex field, N is a real field and λ_i and μ_i, $i=1,2,3,4$, are real parameters. In physical problems like the non-linear development of the modulational instabilities in plasmas [19], $E(x,t)$ represents a field having a high-frequency carrier wave whose amplitude is modulated, and $N(x,t)$ corresponds to the unmodulated low-frequency wave. For stationary solutions for E and N

$$\begin{aligned} E(x,t)&=E(\xi)\exp[\mathrm{i}\{X(x)+T(t)\}] \\ N(x,t)&=N(\xi) \end{aligned} \tag{A3}$$

where $\xi=x-Mt$ is the stationary variable and M is the velocity of the stationary frame. The functions $X(x)$ and $T(t)$ are introduced to account for the shifts in the frequency and wavenumber of the carrier wave due to non-linearities. The real function $E(\xi)$ represents the envelope of the carrier wave.

Substituting (A3) into (A1), we obtain from the imaginary parts

$$\left(2\frac{\mathrm{d}X}{\mathrm{d}x}+\frac{\lambda_1-M}{\lambda_2}\right)\frac{\mathrm{d}E}{\mathrm{d}\xi}+E\frac{\mathrm{d}^2X}{\mathrm{d}x^2}=0 \tag{A4}$$

which has the solution

$$X(x)=\frac{1}{2\lambda_2}(M-\lambda_1)x. \tag{A5}$$

On the other hand, from the real parts, we obtain the following governing equation for the stationary envelope $E(\xi)$:

$$\lambda_2\,\mathrm{d}^2E/\mathrm{d}\xi^2=\lambda E+\lambda_4 NE \tag{A6}$$

where

$$\lambda=\delta+\frac{\lambda_1}{2\lambda_2}(M-\lambda_1)+\frac{1}{4\lambda_2}(M-\lambda_1)^2+\lambda_3 \tag{A7}$$

and $\delta=\mathrm{d}T/\mathrm{d}t$ is the non-linear frequency shift parameter.

Writing (A2) in terms of ξ and integrating, we get

$$\mu_2\,\mathrm{d}^2N/\mathrm{d}\xi^2 = -(M^2+\mu_1)N - \mu_3N^2 + \mu_4E^2. \tag{A8}$$

If the Boussinesq equation (A2) is replaced by a (driven) Korteweg-de Vries equation:

$$\frac{\partial N}{\partial t} + \mu_1\frac{\partial N}{\partial x} + \mu_2\frac{\partial^3 N}{\partial x^3} + \mu_3 N\frac{\partial N}{\partial x} = \mu_4\frac{\partial}{\partial x}(EE^*) \tag{A9}$$

then (A8) is replaced by

$$\mu_2\,\mathrm{d}^2N/\mathrm{d}\xi^2 = (M-\mu_1)N - \tfrac{1}{2}\mu_3N^2 + \mu_4E^2. \tag{A10}$$

Equations (A6) and (A8) (or (A10)) are identical with the generic equations (1) and (2) when the coefficients are suitably redefined.

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